Lecture 14

7.7 - Approximate Integration

More often than not, in practice, we will need to compute an integral $\int_a^b f(x)\,dx$ where we cannot find the antiderivative of $f(x)$ ... through standard means anyway (e.g., using elementary functions). For example, in statistics, you often want to integrate the function

$$\phi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$ (called the normal or Gaussian distribution).

(There is actually a neat trick to integrate this involving double integrals & polar coordinates (c.f., Calc III) which I'll include at the end of the notes!)

Approximation Methods: In Calc I, you learned how to approximate integrals using left & right endpoint Riemann sums as well as the Midpoint Rule:

If f is integrable on $[a,b]$, then:

$$\int_a^b f(x)\,dx \approx M_n = \sum_{j=1}^{n} f(\bar{x}_j)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_j = a + j\Delta x$, $\bar{x}_j = \frac{1}{2}(x_{j-1} + x_j) =$ "midpoint of $[x_{j-1}, x_j]$"

Another approximation method results from averaging left & right endpoint approximations:

Trapezoidal Rule: If f is integrable on $[a,b]$

$$\int_a^b f(x)\,dx \approx T_n = \frac{1}{2}(L_n + R_n)$$

$$= \frac{1}{2}\left(\sum_{j=1}^{n} f(x_{j-1})\Delta x + \sum_{j=1}^{n} f(x_j)\Delta x\right)$$

$$= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

where, again, $\Delta x = \frac{b-a}{n}$, $x_j = a + j\Delta x$

Ex: Use (a) the midpoint rule, and (b) the trapezoidal rule to approximate $\ln(4) = \int_1^4 \frac{1}{x}\,dx$ with $n=6$.

Sol: $\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$

$j=$	0	1	2	3	4	5	6
x_j	1	$3/2$	2	$5/2$	3	$7/2$	4
$\bar{x}_j$	N/A	$5/4$	$7/4$	$9/4$	$11/4$	$13/4$	$15/4$
$f(x_j)$	1	$2/3$	$1/2$	$2/5$	$1/3$	$2/7$	$1/4$
$f(\bar{x}_j)$	N/A	$4/5$	$4/7$	$4/9$	$4/11$	$4/13$	$4/15$

a) $M_6 = \frac{1}{2}\left(\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \frac{4}{13} + \frac{4}{15}\right) = \frac{62024}{45045} \approx 1.3769342$

b) $T_6 = \frac{1}{4}\left(1 + 2\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7}\right) + \frac{1}{4}\right) = \frac{787}{560} \approx 1.4053571$

The actual value is $\ln(4) = 1.3862944$
(to 7th order)

The natural thing to worry about when approximating is how "off" are you from the actual answer, i.e., what is the error. The error for:

- the midpoint rule: $E_M = \int_a^b f(x)\,dx - M_n$
- the trapezoidal rule: $E_T = \int_a^b f(x)\,dx - T_n$

Of course computing the error exactly is usually not possible, so the best we can do is to get bounds on the error. Getting the following formulas requires some clever integration by parts.

Error bounds: If $|f''(x)| \le K$ for $a \le x \le b$, then

$$|E_M| \le \frac{K(b-a)^3}{24n^2} \quad \& \quad |E_T| \le \frac{K(b-a)^3}{12n^2}$$

Ex: For the previous example find ⓐ the error in either case ⓑ what we should choose for n so that the approximation is correct to within $0.000001 = 10^{-6}$.

ⓐ $E_M = \ln(4) - M_6$
$= 0.0093602$

$E_T = \ln(4) - T_6$
$= -0.0190627$

ⓑ $f(x) = \frac{1}{x},\ f'(x) = \frac{-1}{x^2},\ f''(x) = \frac{2}{x^3}$; $|f''(x)| \le 2$ on $[1,4]$

So, $K = 2$.

midpoint: use $n \ge 1500$

$|E_M| \le \frac{2(4-1)^3}{24n^2} = \frac{54}{24n^2} = 10^{-6}$

$\Rightarrow n^2 = \frac{54}{24} \cdot 10^6 = 2.25 \cdot 10^6$

$\Rightarrow n = \sqrt{2.25}\,(10^3) = 1500$

trapezoid: $|E_T| \le \frac{2(4-1)^3}{12 \cdot n^2} = \frac{54}{12n^2} = 10^{-6}$

$\Rightarrow n^2 = \frac{54}{12} \cdot 10^6 = 4.5 \cdot 10^6$

$\Rightarrow n = \sqrt{4.5}\,(10^3) \approx 2.1213 \cdot 10^3 = 2121.3$

Use $n \ge 2122$.

The previous methods were all "first-order" approximations in that the actual function is approximated by lines "ax+b" on each interval. We can take this up a step and try to approximate f(x) to "second-order," i.e., with parabolas "ax^2+bx+c".

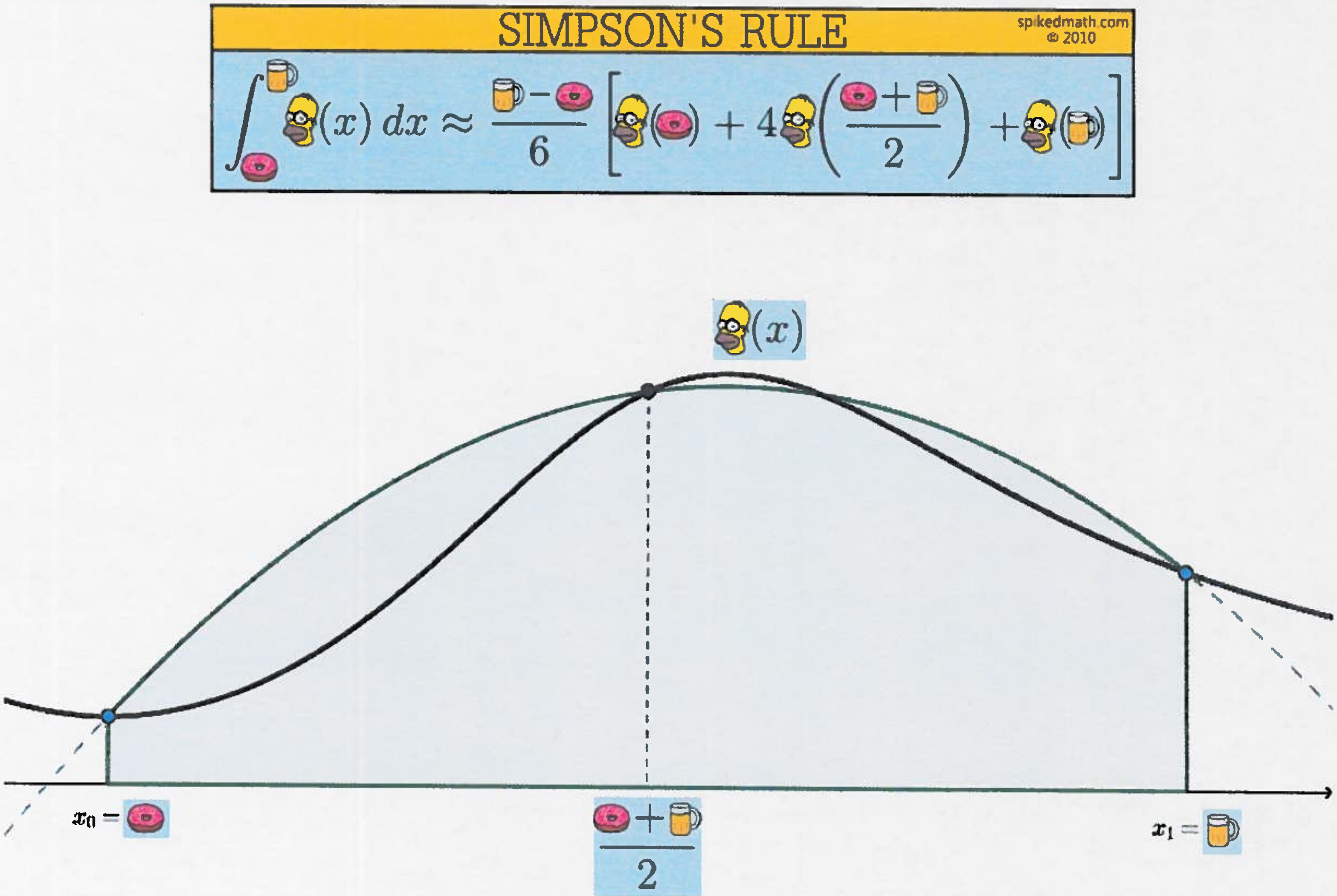
Simpson's Rule:

notice the pattern
1, 4, 2, 4, 2, ⋯, 4, 2, 4, 1

$$\int_a^b f(x)\,dx \approx S_n$$

$$= \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)$$

where $\Delta x = \frac{b-a}{2}$, $x_j = a + j\Delta x$, n even


SIMPSON'S RULE
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(x) dx ≈
6
+ 4
2
(x)
x0 =
2
x1 =

It turns out that

$$\boxed{S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n}$$

Ex: Use Simpson's rule to approximate ln(4) with n=6. What is the error in this approximation?

We have from before: $\Delta x = \frac{1}{2}$

$j=$	0	1	2	3	4	5	6
x_j	1	3/2	2	5/2	3	7/2	4
$f(x_j)$	1	2/3	1/2	2/5	1/3	2/7	1/4

$$S_6 = \frac{1}{6}\left(1+4\left(\frac{2}{3}\right)+2\left(\frac{1}{2}\right)+4\left(\frac{2}{5}\right)+2\left(\frac{1}{3}\right)+4\left(\frac{2}{7}\right)+\frac{1}{4}\right) = \frac{3497}{2520} \approx 1.3876984$$

$$E_S = \ln(4) - S_6 = -0.001404$$

Error Bound for Simpson's Rule

Suppose that $|f^{(4)}(x)| \le K$ for $a \le x \le b$. Then if E_S is the error, we have: $$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

Ex: What should n be so that Simpson's rule is accurate to within $0.000001 = 10^{-6}$ above? $f^{(4)}(x) = \frac{24}{x^5}$, $|f^{(4)}(x)| \le 24$ on $[1,4]$. $K=24$

$$|E_S| \le \frac{24(4-1)^5}{180n^4} = \frac{5832}{180n^4} = 10^{-6} \Rightarrow n^4 = \frac{5832}{180}\cdot 10^6 = 32.4\cdot 10^6 \Rightarrow n \approx 75.44601$$

Use $n \geq 76$.

In practice, we usually do not have actual functions, but a collection of data points. We can use these approximation methods to "integrate the data".

<u>Ex</u>: The table gives the speed of a runner during the first 5 seconds of a race (sampled every half-second by a radar gun). Estimate the distance covered by the runner during these first five seconds.

time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
velocity ($\frac{m}{s}$)	0	4.67	7.34	8.86	9.73	10.22	10.51	10.67	10.78	10.81	10.81

<u>Sol</u>: We've seen that Simpson's rule gives the best approximation; so we'll approximate with that.

From the data we have that:

$a=0,\ b=5,\ n=10,\ \Delta x=0.5$

The table gives us sample points along the runner's velocity function $v(t)$, so:

distance traveled $= \int_0^5 v(t)\,dt \approx S_{10}$

$= \frac{0.5}{3}\left(0+4(4.67)+2(7.34)+4(8.86)+2(9.73)+4(10.22)+2(10.51)+4(10.67)+2(10.78)+4(10.81)+10.81\right)$

$= 44.741\overline{6} \approx 44.74$

The runner covered about 44.74 meters in the first 5 seconds.

Computing $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$

We compute the square of this number:

(just call x by y here)

$$\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$$ (property of double integrals)

$$= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}r^2}\,r\,dr\,d\theta$$ (switching the double integral to polar coords)

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\lim_{R\to\infty} \underbrace{\int_0^{R} r e^{-\frac{1}{2}r^2}\,dr}_{u\text{-sub}:\ u=-\frac{1}{2}r^2}\right)d\theta$$ (dealing with improper integrals)

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\lim_{R\to\infty}\int_0^{-\frac{1}{2}R^2} -e^{u}\,du\right)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(\lim_{R\to\infty}\left(\overbrace{\cancel{-e^{-\frac{1}{2}R^2}}}^{\to 0 \text{ as } R\to\infty} + e^{0}\right)\right)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

$$\Rightarrow \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx\right)^2 = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx = 1$$